A continuous/discrete fractional Noether's theorem

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Abstract

We prove a fractional Noether's theorem for fractional Lagrangian systems invariant under a symmetry group both in the continuous and discrete cases. This provides an explicit conservation law (first integral) given by a closed formula which can be algorithmically implemented. In the discrete case, the conservation law is moreover computable in a finite number of steps.

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1 Introduction

A conservation law for a continuous or a discrete dynamical system is a function which is constant on each solution. Precisely, let us denote by $q = (q(t))_{t\in[a,b]}$ (resp. $Q = (Q_k)_{k=0,\dots,N}$) the solutions of a continuous (resp. discrete) system taking values in a set denoted by E. Then, a function $C : E \to \mathbb{R}$ is a conservation law (or first integral) if, for any solution q (resp. Q), it exists a real constant c such that:

$$C\big(q(t)\big) = c,\ \forall t \in [a,b]. \quad (\text{resp. } C(Q_k) = c,\ \forall k = 0,\dots,N.) \tag{1.1}$$

Conservation laws are generally associated to some physical quantities like total energy or angular momentum in mechanical systems and sometimes, they can be used to reduce or integrate by quadrature the equation.

In this paper, we study the existence of classical conservation laws for continuous or discrete fractional Lagrangian systems introduced in [1] and [4] respectively. We follow the usual strategy of the Noether's theorem which provides explicit conservation laws for classical Lagrangian systems invariant under symmetries. In particular, we prove a fractional Noether's theorem both in the continuous and discrete cases providing explicit formula of the conservation law. This formula is algorithmic so that arbitrary hight order approximations can be computed. In the discrete case, the algorithm is moreover finite.

Previous results in this direction have been obtained by several authors ([3],[5],[6]). However, in each of these papers, the conservation law is not explicitly derived in term of the symmetry group and the Lagrangian but implicitly defined by functional relations ([5],[6]) or integral relations [3].

The paper is organized as follows. In Section 2, we first remind classical definitions and results concerning fractional derivatives and fractional Lagrangian systems. Then, we prove a transfer formula leading to the statement of the fractional Noether's theorem based on a result of Cresson in [5]. Finally, we give some remarks concerning the previous studies on the subject and we give some precisions on our result. In Section 3, we first remind the definition of the discrete fractional Lagrangian systems in the sense of [4]. Then, introducing the notion of discrete symmetry for these systems, we state the discrete fractional Noether's theorem. We conclude with some remarks and a numerical implementation.

2 A fractional Noether's theorem

In this paper, we consider fractional differential systems in $\mathbb{R}^d$ where $d \in \mathbb{N}^*$ is the dimension. The trajectories of these systems are curves q in $\mathcal{C}^0([a,b],\mathbb{R}^d)$ where $a < b$ are two reals. Let $0 < \alpha \leq 1$ denote a fractional order.

2.1 Reminder about fractional Lagrangian systems

We first review classical definitions extracted from [8, 10, 11] of the fractional derivatives of Riemann-Liouville. Let Γ be the Gamma function of Euler and f be an element of $\mathcal{C}^1([a,b],\mathbb{R}^d)$. The fractional left (respectively right) integral of order α with inferior limit a (respectively superior limit b) of f is given by:

$$\forall t \in]a,b],\ I^{\alpha}_{-} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-y)^{\alpha-1} f(y) dy, \tag{2.1}$$

respectively:

$$\forall t \in [a,b[,\ I^{\alpha}_{+} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (y-t)^{\alpha-1} f(y) dy. \tag{2.2}$$

Then, the fractional left (respectively right) derivative of order α with inferior limit a (respectively superior limit b) is given by:

$$\forall t \in]a,b],\ D^{\alpha}_{-} f(t) = \frac{d}{dt}\Big(I^{1-\alpha}_{-} f\Big)(t), \tag{2.3}$$

respectively:

$$\forall t \in [a,b[,\ D^{\alpha}_{+} f(t) = -\frac{d}{dt}\Big(I^{1-\alpha}_{+} f\Big)(t). \tag{2.4}$$

A fractional Lagrangian functional is an application defined by:

$$\begin{array}{rccc} \mathcal{L}^{\alpha}: & \mathcal{C}^2([a,b],\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \displaystyle\int_a^b L\big(q(t), D^{\alpha}_{-} q(t), t\big) dt, \end{array} \tag{2.5}$$

where L is a Lagrangian *i.e.* a $\mathcal{C}^2$ application defined by:

$$\begin{array}{rccc} L: & \mathbb{R}^d \times \mathbb{R}^d \times [a,b] & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & L(x,v,t). \end{array} \tag{2.6}$$

It is well-known that extremals of a fractional Lagrangian functional can be characterized by a variational principle as solutions of a fractional differential system, see [1]. Precisely, q is a

critical point of $\mathcal{L}^\alpha$ if and only if q is solution of the fractional Euler-Lagrange equation given by:

$$\frac{\partial L}{\partial x}\big(q, D_-^\alpha q, t\big) + D_+^\alpha \left(\frac{\partial L}{\partial v}\big(q, D_-^\alpha q, t\big) \right) = 0. \tag{EL$^\alpha$}$$

A dynamical system governed by an equation shaped as (EL$^\alpha$) is called fractional Lagrangian system.

For $\alpha = 1$, we have $D_-^1 = -D_+^1 = d/dt$ and (EL$^\alpha$) coincides with the classical Euler-Lagrange equation (see [2]).

2.2 Symmetries

We first review the definition of a one parameter group of diffeomorphisms:

Definition 1. *For any real s, let $\phi(s,\cdot): \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a diffeomorphism. Then, $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$ is a one parameter group of diffeomorphisms if it satisfies*

1. $\phi(0,\cdot) = Id_{\mathbb{R}^d}$,
2. $\forall s, s' \in \mathbb{R},\ \phi(s,\cdot) \circ \phi(s',\cdot) = \phi(s+s',\cdot)$,
3. *ϕ is of class $\mathcal{C}^2$.*

Classical examples of one parameter groups of diffeomorphisms are given by translations and rotations.

The action of a one parameter group of diffeomorphisms on a Lagrangian allows to define the notion of a symmetry for a fractional Lagrangian system:

Definition 2. *Let $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms and let L be a Lagrangian. L is said to be D_-^α-invariant under the action of Φ if it satisfies:*

$$\forall q \text{ solution of } (\mathrm{EL}^\alpha),\ \forall s \in \mathbb{R},\ L\Big(\phi(s,q), D_-^\alpha\big(\phi(s,q)\big), t\Big) = L\big(q, D_-^\alpha q, t\big). \tag{2.7}$$

2.3 A fractional Noether's theorem

Cresson [5] and Torres *et al.* [6] proved the following result:

Lemma 3. *Let L be a Lagrangian D_-^α-invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$. Then, the following equality holds for any solutions q of (EL$^\alpha$):*

$$D_-^\alpha\left(\frac{\partial \phi}{\partial s}(0,q)\right) \cdot \frac{\partial L}{\partial v}(q, D_-^\alpha q, t) - \frac{\partial \phi}{\partial s}(0,q) \cdot D_+^\alpha\left(\frac{\partial L}{\partial v}(q, D_-^\alpha q, t)\right) = 0. \tag{2.8}$$

In the classical case $\alpha = 1$, the classical Leibniz formula allows to rewrite (2.8) as the derivative of a product. Precisely, from Lemma 3, it leads to the classical Noether's theorem given by:

Theorem 4 (Classical Noether's theorem)**.** *Let L be a Lagrangian d/dt-invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$. Then, the following equality holds for any solutions q of* (EL^1)*:*

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial s}(0,q)\cdot\frac{\partial L}{\partial v}(q,\dot{q},t)\right) = 0, \tag{2.9}$$

where $\dot{q}$ is the classical derivative of q, i.e. dq/dt.

Theorem 4 provides an explicit constant of motion for any classical Lagrangian systems admitting a symmetry. In the fractional case, such a simple formula allowing to rewrite (2.8) as a total derivative with respect to t is not known. The next theorem provides such a formula in an explicit form:

Theorem 5 (Transfer formula)**.** *Let $f, g \in \mathcal{C}^\infty([a,b],\mathbb{R}^d)$ assuming the following condition (C):*

the sequences of functions $(I_-^{p-\alpha} f \cdot g^{(p)})_{p\in\mathbb{N}^*}$ and $(f^{(p)} \cdot I_+^{p-\alpha} g)_{p\in\mathbb{N}^*}$ converge uniformly to 0 on $[a,b]$.

Then, the following equality holds:

$$D_-^\alpha f \cdot g - f \cdot D_+^\alpha g = \frac{d}{dt}\left[\sum_{r=0}^{\infty}(-1)^r I_-^{r+1-\alpha} f \cdot g^{(r)} + f^{(r)} \cdot I_+^{r+1-\alpha} g\right]. \tag{2.10}$$

Proof. Let $f, g \in \mathcal{C}^\infty([a,b],\mathbb{R}^d)$. By induction and using the classical Leibniz formula at each step, we prove that for any $p \in \mathbb{N}^*$, the following equalities both hold:

$$D_-^\alpha f \cdot g = (-1)^p I_-^{p-\alpha} f \cdot g^{(p)} + \frac{d}{dt}\left[\sum_{r=0}^{p-1}(-1)^r I_-^{r+1-\alpha} f \cdot g^{(r)}\right] \tag{2.11}$$

and

$$-f \cdot D_+^\alpha g = f^{(p)} \cdot I_+^{p-\alpha} g + \frac{d}{dt}\left[\sum_{r=0}^{p-1} f^{(r)} \cdot I_+^{r+1-\alpha} g\right]. \tag{2.12}$$

For any $p \in \mathbb{N}^*$, we denote by u_p the following function:

$$u_p := \sum_{r=0}^{p-1}(-1)^r I_-^{r+1-\alpha} f \cdot g^{(r)}. \tag{2.13}$$

According to (2.11), for any $p \in \mathbb{N}^*$, $\dot{u}_p = D_-^\alpha f \cdot g - (-1)^p I_-^{p-\alpha} f \cdot g^{(p)}$. Hence, the assumption made on the sequence of functions $(I_-^{p-\alpha} f \cdot g^{(p)})_{p\in\mathbb{N}^*}$ implies that the sequence $(\dot{u}_p)_{p\in\mathbb{N}^*}$ converges uniformly to $D_-^\alpha f \cdot g$ on $[a,b]$. Moreover, let us note that $(u_p)_{p\in\mathbb{N}^*}$ point-wise converges in $t = a$. Finally, we can conclude that the sequence $(u_p)_{p\in\mathbb{N}^*}$ converges uniformly on $[a,b]$ to a function u equal to

$$u := \sum_{r=0}^{\infty}(-1)^r I_-^{r+1-\alpha} f \cdot g^{(r)} \tag{2.14}$$

and satisfying $\dot{u} = D_-^\alpha f \cdot g$. Similarly, one can prove that:

$$\frac{d}{dt}\left[\sum_{r=0}^{\infty} f^{(r)} \cdot I_+^{r+1-\alpha} g\right] = -f \cdot D_+^\alpha g.$$

□

Thus, combining Lemma 3 and this transfer formula, we prove:

Theorem 6 (A fractional Noether's theorem). *Let L be a Lagrangian D_-^α-invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$. Let q be a solution of* (EL$^\alpha$) *and let f and g denote:*

$$f = \frac{\partial \phi}{\partial s}(0,q) \quad \textit{and} \quad g = \frac{\partial L}{\partial v}(q, D_-^\alpha q, t). \tag{2.15}$$

If f and g satisfy the condition (C), then the following equality holds:

$$\frac{d}{dt}\left[\sum_{r=0}^{\infty} (-1)^r I_-^{r+1-\alpha} f \cdot g^{(r)} + f^{(r)} \cdot I_+^{r+1-\alpha} g\right] = 0. \tag{2.16}$$

This theorem provides an explicit algorithmic way to compute a constant of motion for any fractional Lagrangian systems admitting a symmetry. An arbitrary closed approximation of this quantity can be obtained with a truncature of the infinite sum.

Previous results in this direction have been obtained by several authors. We give some details and comments in the following.

- In [6], Torres *et al.* define a *fractional conservation law* for fractional Lagrangian systems. Precisely, denoting by D_0^α the bilinear operator

$$\forall f, g \in \mathcal{C}^1([a,b],\mathbb{R}^d),\ D_0^\alpha(f,g) = D_-^\alpha f \cdot g - f \cdot D_+^\alpha g, \tag{2.17}$$

  they give the following definition: a function $\begin{array}{rccc} C: & \mathbb{R}^d \times \mathbb{R}^d \times [a,b] & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & C(x,v,t) \end{array}$ is a *fractional-conserved quantity* of a fractional system if it is possible to write C in the form $C(x,v,t) = C^1(x,v,t) \cdot C^2(x,v,t)$ with $D_0^\alpha\big(C^1(q,D_-^\alpha q,t), C^2(q,D_-^\alpha q,t)\big) = 0$ for any solution q of the considered system. They deduce easily that $\partial\phi/\partial s(0,x) \cdot \partial L/\partial v(x,v,t)$ is a fractional-conserved quantity of (EL$^\alpha$). Two important difficulties appear:

  - Although all these notions and results coincide with the classical ones when $\alpha = 1$, we do not have that $D_0^\alpha(f,g) = 0$ implies $f \cdot g$ is constant.
  - The decomposition $C = C^1 \cdot C^2$ is not unique and it leads to many issues. Indeed, the scalar product is commutative and then $C = C^1 \cdot C^2 = C^2 \cdot C^1$. Nevertheless, the operator D_0^α is not symmetric! Hence, in the result of Torres *et al*, we have:

$$C(x,v,t) = \frac{\partial \phi}{\partial s}(0,x) \cdot \frac{\partial L}{\partial v}(x,v,t) = \frac{\partial L}{\partial v}(x,v,t) \cdot \frac{\partial \phi}{\partial s}(0,x) \tag{2.18}$$

    is a fractional-conserved quantity of (EL$^\alpha$) but we do not have:

$$D_0^\alpha\left(\frac{\partial L}{\partial v}(q, D_-^\alpha q, t), \frac{\partial \phi}{\partial s}(0,q)\right) = 0. \tag{2.19}$$

- In [3], Atanackovič *et al.* were interested by symmetries modifying also the time variable t. They finally obtained in [3, Theorem 15] a formulation of constants of motion for fractional Lagrangian systems admitting such symmetries. Nevertheless, if we consider a symmetry Φ which does not modify the time variable, the result of Atanackovič *et al.* is the following: for any solutions q of (EL^α), let $f = \partial\phi/\partial s(0,q)$ and $g = \partial L/\partial v(q, D^\alpha_- q, t)$, the following element $t \longmapsto \int_a^t D^\alpha_- f \cdot g - f \cdot D^\alpha_+ g \, dy$ is constant on $[a,b]$. This result is then unsatisfactory because the constant of motion is not explicit in contrary to our result.

2.4 Concerning the condition (C)

Condition (C) could seem strong or too particular. In order to make it more concrete and understand what classes of functions satisfy (C), we give the two following sufficient conditions:

Proposition 7. *Let $f, g \in \mathcal{C}^\infty([a,b], \mathbb{R}^d)$.*

1. *If f and g satisfy the two following conditions:*
$$\max_{t\in[a,b]} \left(\frac{(b-t)^{p-1}}{(p-1)!} |f^{(p)}(t)| \right) \underset{p\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \max_{t\in[a,b]} \left(\frac{(t-a)^{p-1}}{(p-1)!} |g^{(p)}(t)| \right) \underset{p\to\infty}{\longrightarrow} 0 \tag{2.20}$$
then f and g satisfy the condition (C).

2. *If there exists $M > 0$ such that:*
$$\forall p \in \mathbb{N}^*, \ \forall t \in [a,b], \ |f^{(p)}(t)| \leq M \quad \text{and} \quad |g^{(p)}(t)| \leq M \tag{2.21}$$
then f and g satisfy the condition (C).

Proof. 1. Let us prove that the sequence of functions $(I^{p-\alpha}_- f \cdot g^{(p)})_{p\in\mathbb{N}^*}$ converges uniformly to 0. The proof is similar for the sequence $(f^{(p)} \cdot I^{p-\alpha}_- g)_{p\in\mathbb{N}^*}$. Let us denote $M = \max_{t\in[a,b]}(|f(t)|)$. For any $p \in \mathbb{N}^*$ and any $t \in [a,b]$:
$$\begin{aligned} \left| I^{p-\alpha}_- f(t) g^{(p)}(t) \right| &= \left| \frac{g^{(p)}(t)}{\Gamma(p-\alpha)} \int_a^t (t-y)^{p-\alpha-1} f(y) dy \right| \\ &\leq M \frac{(t-a)^{p-\alpha}}{\Gamma(p+1-\alpha)} |g^{(p)}(t)|. \end{aligned} \tag{2.22}$$
Finally, as $\Gamma(p+1-\alpha) \geq (p-1)!\,\Gamma(2-\alpha)$:
$$\begin{aligned} \left| I^{p-\alpha}_- f(t) g^{(p)}(t) \right| &\leq M \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \frac{(t-a)^{p-1}}{(p-1)!} |g^{(p)}(t)| \\ &\leq M \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \frac{(t-a)^{p-1}}{(p-1)!} |g^{(p)}(t)|, \end{aligned} \tag{2.23}$$
which concludes the proof.
2. It follows from the first result. ☐

For example, if f is polynomial and g is the exponential function, one can prove that f and g satisfy the point 2 of Proposition 7 and then the condition (C). If $g(t) = 1/t$ on $[a,b]$ with $a > 0$, one can notice that the point 2 of Proposition 7 is not satisfied anymore. Nevertheless, the point 1 is true and consequently the functions f and g satisfy the condition (C). In general, every couple of analytic functions satisfy condition (C) and consequently (2.10) is true.

3 A discrete fractional Noether's theorem

We study the existence of discrete conservation laws for discrete fractional Lagrangian systems in the sense of [4]. Using the same strategy as in the continuous case, we introduce the notion of discrete symmetry and prove a discrete fractional Noether's theorem providing an *explicit computable* discrete constant of motion.

3.1 Reminder about discrete fractional Lagrangian systems

We follow the definition of discrete fractional Lagrangian systems given in [4] to which we refer for more details.

We denote by $N \in \mathbb{N}^*$, by $h = (b-a)/N$ the step size of the discretization and by $\tau = (t_k)_{k=0,...,N}$ the following partition of $[a,b]$:

$$\forall k = 0, ..., N, \ t_k = a + kh. \tag{3.1}$$

Let us denote by Δ^α_- and Δ^α_+ the following discrete analogous of D^α_- and D^α_+ respectively:

$$\begin{array}{rccl} \Delta^\alpha_- : & (\mathbb{R}^d)^{N+1} & \longrightarrow & (\mathbb{R}^d)^N \\ & Q & \longmapsto & \left(\dfrac{1}{h^\alpha} \displaystyle\sum_{r=0}^{k} \alpha_r Q_{k-r} \right)_{k=1,..,N} , \end{array} \tag{3.2}$$

$$\begin{array}{rccl} \Delta^\alpha_+ : & (\mathbb{R}^d)^{N+1} & \longrightarrow & (\mathbb{R}^d)^N \\ & Q & \longmapsto & \left(\dfrac{1}{h^\alpha} \displaystyle\sum_{r=0}^{N-k} \alpha_r Q_{k+r} \right)_{k=0,..,N-1} , \end{array} \tag{3.3}$$

where the elements $(\alpha_r)_{r\in\mathbb{N}}$ are defined by $\alpha_0 = 1$ and

$$\forall r \in \mathbb{N}^*, \ \alpha_r = \frac{(-\alpha)(1-\alpha)...(r-1-\alpha)}{r!}. \tag{3.4}$$

These discrete fractional operators are approximations of the continuous ones. We refer to [10] for more details.

A discrete fractional Euler-Lagrange equation, of unknown $Q \in (\mathbb{R}^d)^{N+1}$, is defined by:

$$\frac{\partial L}{\partial x}(Q, \Delta^\alpha_- Q, \tau) + \Delta^\alpha_+ \left(\frac{\partial L}{\partial v}(Q, \Delta^\alpha_- Q, \tau) \right) = 0, \tag{EL$^\alpha_h$}$$

where L is a Lagrangian. In such a case, we speak of a discrete Lagrangian system.

The terminology is justified by the fact that solutions of (EL$^\alpha_h$) correspond to discrete critical points of the discrete fractional Lagrangian functional defined by:

$$\begin{array}{rccl} \mathcal{L}^\alpha_h : & (\mathbb{R}^d)^{N+1} & \longrightarrow & \mathbb{R} \\ & Q & \longmapsto & h \displaystyle\sum_{k=1}^{N} L\big(Q_k, (\Delta^\alpha_- Q)_k, t_k\big). \end{array} \tag{3.5}$$

We refer to [4] for a proof.

In the case $\alpha = 1$, the discrete operators $\Delta^1_\pm$ correspond to the implicit and explicit Euler approximations of d/dt and (EL^α_h) is just the discrete Euler-Lagrange equation obtained in [7, 9].

3.2 Discrete symmetries

Here again, a discrete symmetry of a discrete fractional Lagrangian system is based on the action of a one parameter group of transformations on the associated Lagrangian:

Definition 8. *Let $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms and let L be a Lagrangian. L is said to be Δ^α_--invariant under the action of Φ if it satisfies:*

$$\forall Q \text{ solution of } (\mathrm{EL}^\alpha_h),\ \forall s \in \mathbb{R},\ L\Big(\phi(s,Q), \Delta^\alpha_-\big(\phi(s,Q)\big), \tau\Big) = L\big(Q, \Delta^\alpha_- Q, \tau\big). \tag{3.6}$$

Then, we can prove the following discrete version of Lemma 3:

Lemma 9. *Let L be a Lagrangian Δ^α_--invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s,\cdot)\}_{s\in\mathbb{R}}$. Then, the following equality holds for any solutions Q of (EL^α_h):*

$$\Delta^\alpha_-\left(\frac{\partial\phi}{\partial s}(0,Q)\right)\cdot\frac{\partial L}{\partial v}(Q,\Delta^\alpha_- Q,\tau) - \frac{\partial\phi}{\partial s}(0,Q)\cdot\Delta^\alpha_+\left(\frac{\partial L}{\partial v}(Q,\Delta^\alpha_- Q,\tau)\right) = 0. \tag{3.7}$$

Proof. This proof is a direct adaptation to the discrete case of the proof of Lemma 3. Let $Q \in (\mathbb{R}^d)^{N+1}$ be a solution of (EL^α_h). Let us differentiate equation (3.6) with respect to s and invert the operators Δ^α_- and $\partial/\partial s$. We finally obtain for any $s \in \mathbb{R}$ and any $k \in \{1,...,N-1\}$:

$$\begin{aligned}\Delta^\alpha_-\left(\frac{\partial\phi}{\partial s}(s,Q)\right)_k\cdot\frac{\partial L}{\partial v}\Big(\phi(s,Q_k),\Delta^\alpha_-\big(\phi(s,Q_k)\big),t_k\Big)&\\ +\frac{\partial L}{\partial x}\Big(\phi(s,Q_k),\Delta^\alpha_-\big(\phi(s,Q_k)\big),t_k\Big)\cdot\frac{\partial\phi}{\partial s}(s,Q_k) &= 0.\end{aligned} \tag{3.8}$$

Since $\phi(0,\cdot) = Id_{\mathbb{R}^d}$ and Q is solution of (EL^α_h), taking $s = 0$ in (3.8) leads to (3.7). $\square$

3.3 A discrete fractional Noether's theorem

Let us remind that our aim is to express explicitly a discrete constant of motion for discrete fractional Lagrangian systems admitting a discrete symmetry. As in the continuous case, our result is based on Lemma 9. Let us note that the following implication holds:

$$\forall F \in \mathbb{R}^{N+1},\ \Delta^1_- F = 0 \Longrightarrow \exists c \in \mathbb{R},\ \forall k = 0,...,N,\ F_k = c. \tag{3.9}$$

Namely, if the discrete derivative of F vanishes, then F is constant. Our aim is then to write (3.7) as a discrete derivative (*i.e.* as Δ^1_- of an explicit quantity).

We first introduce some notations and definitions. The shift operator denoted by σ is defined by

$$\begin{array}{rccl}\sigma: & (\mathbb{R}^d)^{N+1} & \longrightarrow & (\mathbb{R}^d)^{N+1}\\ & Q & \longmapsto & \sigma(Q) = (Q_{k+1})_{k=0,...,N}\end{array} \tag{3.10}$$

with the convention $Q_{N+1} = 0$. We also introduce the following square matrices of length $(N+1)$. First, $A_1 = -Id_{N+1}$ and then, for any $r \in \{2, ..., N-1\}$, the square matrices $A_r \in \mathcal{M}_{N+1}$ defined by:

$$\forall i, j = 0, ..., N, \ (A_r)_{i,j} = \begin{cases} 0 & \text{if} \quad i = 0 \\ \delta_{\{j=0\}}\delta_{\{r \leq i\}} - \delta_{\{0 \leq i-j \leq r-1\}}\delta_{\{1 \leq j \leq N-r\}} & \text{if} \quad 1 \leq i \leq N-1 \\ (A_r)_{N-1,j} & \text{if} \quad i = N \end{cases} \tag{3.11}$$

where δ is the Kronecker symbol.

For example, for $N = 5$, the matrices A_r are given by: $A_1 = -Id_6$ and

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Considering these previous elements, we state the following result:

Theorem 10 (A discrete fractional Noether's theorem)**.** *Let L be a Lagrangian Δ^{α}_{-}-invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$. Then, the following equality holds for any solutions Q of* (EL^{α}_{h})*:*

$$\Delta^{1}_{-} \left[\sum_{r=1}^{N-1} \alpha_r A_r \left(\frac{\partial \phi}{\partial s}(0, Q) \cdot \sigma^r \Big(\frac{\partial L}{\partial v}(Q, \Delta^{\alpha}_{-} Q, \tau) \Big) \right) \right] = 0. \tag{3.12}$$

Combining (3.9) and (3.12), Theorem 10 provides a constant of motion for discrete fractional Lagrangian systems admitting a symmetry. Let us note that the discrete conservation law is not only explicit but computable in finite time. For an example, we give a numerical test in the next section.

Before giving the proof, we remark that Theorem 10 takes a particular simple expression when $\alpha = 1$. Indeed, since $\alpha_r = 0$ for any $r \geq 2$, $\alpha_1 = -1$ and $A_1 = -Id_{N+1}$ in this case, we obtain:

Theorem 11 (Discrete classical Noether's theorem)**.** *Let L be a Lagrangian Δ^{1}_{-}-invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$. Then, the following equality holds for any solutions Q of* (EL^{1}_{h})*:*

$$\Delta^{1}_{-} \left[\frac{\partial \phi}{\partial s}(0, Q) \cdot \sigma \left(\frac{\partial L}{\partial v}(Q, \Delta^{1}_{-} Q, \tau) \right) \right] = 0. \tag{3.13}$$

This result is a reformulation of the discrete Noether's theorem proved in [7, 9]. It corresponds to our Lemma 9 with $\alpha = 1$ using the following discrete Leibniz formula:

$$\forall F, G \in (\mathbb{R}^d)^{N+1}, \ \Delta^{1}_{-} \big(F \cdot \sigma(G) \big) = \Delta^{1}_{-} F \cdot G - F \cdot \Delta^{1}_{+} G. \tag{3.14}$$

Proof of Theorem 10. According to Lemma 9, equation (3.7) holds for any $k = 1, ..., N-1$. Let $F = \partial\phi/\partial s(0, Q)$ and $G = \partial L/\partial v(Q, \Delta_-^\alpha Q, \tau)$. Let us multiply (3.7) by h^α and obtain the following equality for any $k = 1, ..., N-1$:

$$\left(\sum_{r=0}^{k} \alpha_r F_{k-r}\right) \cdot G_k - F_k \cdot \left(\sum_{r=0}^{N-k} \alpha_r G_{k+r}\right) = 0, \tag{3.15}$$

and as $\alpha_0 = 1$:

$$\alpha_1 (F_{k-1} \cdot G_k - F_k \cdot G_{k+1})_k + \left(\sum_{r=2}^{k} \alpha_r F_{k-r}\right) \cdot G_k - F_k \cdot \left(\sum_{r=2}^{N-k} \alpha_r G_{k+r}\right) = 0, \tag{3.16}$$

then:

$$\alpha_1 \Delta_-^1 \big(F \cdot \sigma(G)\big)_k = \frac{1}{h} \left[\left(\sum_{r=2}^{k} \alpha_r F_{k-r}\right) \cdot G_k - F_k \cdot \left(\sum_{r=2}^{N-k} \alpha_r G_{k+r}\right)\right]. \tag{3.17}$$

Let us denote J_k the right term of (3.17) for any $k = 1, ..., N-1$. We are going to write J_k as the discrete derivative of its discrete anti-derivative: it corresponds to the discrete version of the method of Atanackovìc in [3]. For any $k = 1, ..., N-1$, we obtain $J_k = (\Delta_-^1 H)_k$ where $H_i := h \sum_{j=1}^{i} J_j$ for any $i = 0, ..., N$ with the convention $H_0 = J_N = 0$. Let us provide an explicit formulation of the element $H = (H_i)_{i=0,...,N}$.

- *Case* $1 \leq i \leq N-1$:

$$H_i = \sum_{j=1}^{i} \left[\sum_{r=2}^{j} \alpha_r F_{j-r} \cdot G_j - \sum_{r=2}^{N-j} \alpha_r F_j \cdot G_{j+r}\right] = \sum_{j=2}^{i} \sum_{r=2}^{j} \alpha_r F_{j-r} \cdot G_j - \sum_{j=1}^{i} \sum_{r=2}^{N-j} \alpha_r F_j \cdot G_{j+r}. \tag{3.18}$$

As $\sum_{j=2}^{i} \sum_{r=2}^{j} = \sum_{r=2}^{i} \sum_{j=r}^{i}$, we have:

$$\sum_{j=2}^{i} \sum_{r=2}^{j} \alpha_r F_{j-r} \cdot G_j = \sum_{r=2}^{i} \sum_{j=r}^{i} \alpha_r F_{j-r} \cdot G_j = \sum_{r=2}^{i} \sum_{j=0}^{i-r} \alpha_r F_j \cdot G_{j+r} = \sum_{r=2}^{i} \sum_{j=0}^{i-r} \alpha_r \big(F \cdot \sigma^r(G)\big)_j. \tag{3.19}$$

Then, as $\sum_{j=1}^{i} \sum_{r=2}^{N-j} = \sum_{j=1}^{i} \sum_{r=2}^{N-i} + \sum_{j=1}^{i} \sum_{r=N+1-i}^{N-j} = \sum_{r=2}^{N-i} \sum_{j=1}^{i} + \sum_{r=N+1-i}^{N-1} \sum_{j=1}^{N-r}$:

$$H_i = \sum_{r=2}^{i} \sum_{j=0}^{i-r} \alpha_r \big(F \cdot \sigma^r(G)\big)_j - \sum_{r=2}^{N-i} \sum_{j=1}^{i} \alpha_r \big(F \cdot \sigma^r(G)\big)_j - \sum_{r=N+1-i}^{N-1} \sum_{j=1}^{N-r} \alpha_r \big(F \cdot \sigma^r(G)\big)_j. \tag{3.20}$$

Finally, we have:

$$H_i = \sum_{r=2}^{N-1} \sum_{j=0}^{N} \alpha_r A_r(i,j) \big(F \cdot \sigma^r(G)\big)_j, \tag{3.21}$$

where the elements $\big(A_r(i,j)\big)$ are defined for $r = 2, ..., N-1$ and $j = 0, ..., N$ as the real coefficients in front of $\alpha_r \big(F \cdot \sigma^r(G)\big)_j$. Our aim is then to express the values of these elements. From (3.20), we have for any $r = 2, ..., N-1$ and any $j = 0, ..., N$:

$$A_r(i,j) = \delta_{\{r \leq i\}} \delta_{\{0 \leq j \leq i-r\}} - \delta_{\{r \leq N-i\}} \delta_{\{1 \leq j \leq i\}} - \delta_{\{N+1-i \leq r\}} \delta_{\{1 \leq j \leq N-r\}}. \tag{3.22}$$

For example, for $j=0$, we have:

$$\forall r=2,...,N-1,\ A_r(i,0)=\delta_{\{r\leq i\}}. \tag{3.23}$$

Let $r\in\{2,...,N-1\}$ and $j\in\{1,...,N\}$. Let us prove that:

$$A_r(i,j)=-\delta_{\{0\leq i-j\leq r-1\}}\delta_{\{j\leq N-r\}}=-\delta_{\{j\leq i\}}\delta_{\{i-j\leq r-1\}}\delta_{\{j\leq N-r\}}. \tag{3.24}$$

Let us see the four following cases:

- if $j>i$, then $j>i-r$. Moreover, if $N+1-i\leq r$ then $j>N-r$. In this case, $A_r(i,j)=0-0-0=0$.
- if $i-j>r-1$ then $r\leq i$, $j\leq i-r$, $j\leq i$ and $j\leq N-r$. Then, $A_r(i,j)=1-1-0=0$ or $A_r(i,j)=1-0-1=0$ depending on $r\leq N-i$ or $N+1-i\leq r$. Finally, in this case, $A_r(i,j)=0$.
- if $j>N-r$ then $j>i-r$. Moreover, if $r\leq N-i$ then $j>i$. In this case, $A_r(i,j)=0-0-0=0$.
- if $j\leq i$, $i-j\leq r-1$ and $j\leq N-r$, then $i-r<j$. In this case, $A_r(i,j)=0-1-0=-1$ or $A_r(i,j)=0-0-1=-1$ depending on $r\leq N-i$ or $N+1-i\leq r$. Finally, in this case, $A_r(i,j)=-1$.

Consequently, (3.24) holds for any $r\in\{2,...,N-1\}$ and any $j\in\{1,...,N\}$. Finally, from (3.24) and (3.23), we have:

$$\forall r=2,...,N-1,\ \forall j=0,...,N,\ A_r(i,j)=\delta_{\{j=0\}}\delta_{\{r\leq i\}}-\delta_{\{0\leq i-j\leq r-1\}}\delta_{\{1\leq j\leq N-r\}}. \tag{3.25}$$

• *Case* $i=0$ or $i=N$. As $H_0=0$, for any $r=2,...,N-1$ and any $j=0,...,N$, we define $A_r(0,j)=0$. As $H_N=H_{N-1}$, for any $r=2,...,N-1$ and any $j=0,...,N$, we define $A_r(N,j)=A_r(N-1,j)$.

Hence, for any $r=2,...,N-1$, the elements $\big(A_r(i,j)\big)$ are defined for $i=0,...,N$ and $j=0,...,N$. Then, we denote by A_r the matrix $\big(A_r(i,j)\big)_{0\leq i,j\leq N}\in\mathcal{M}_{N+1}$. Finally, from (3.17), we have proved that $\Delta^1_-\big(-\alpha_1 F\cdot\sigma(G)+H\big)=0$ where $H=\sum_{r=2}^{N-1}\alpha_r A_r\big(F\cdot\sigma^r(G)\big)$. Finally, denoting $A_1=-Id_{N+1}\in\mathcal{M}_{N+1}$, we conclude the proof. □

Remark 12. *Using the discrete Leibniz formula given by* (3.14)*, another choice in order to give a discrete version of* (2.16) *from Lemma 9 would be to apply the discrete version of the method used in Section 2.3. Nevertheless, we would encounter many numerical difficulties. Firstly, such a method would imply the use of the operator* Δ^p_- *but this operator approximates the operator* $(d/dt)^p$ *only for* t_k *with* $k\geq p$*. For the* p *first terms, we obtain in general a numerical blow up. Secondly, the use of operator* Δ^p_- *implies the use of* h^{-p}*. Hence, for a large enough* p*, we exceed the machine precision.*

3.4 Numerical test

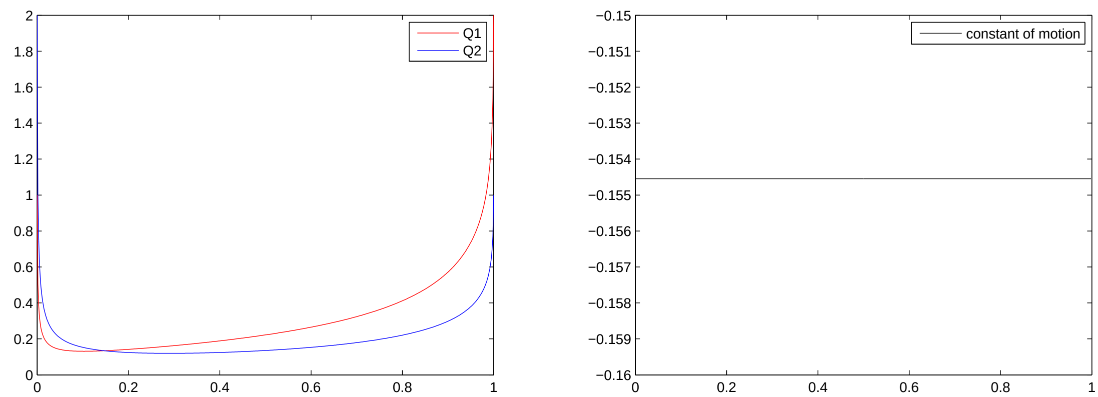
We consider the classical bi-dimensional example ($d = 2$) of the quadratic Lagrangian $L(x, v, t) = (x^2 + v^2)/2$ with $[a, b] = [0, 1]$ and $\alpha = 1/2$. Then, L is Δ^α_--invariant under the action of the rotations given by:

$$\begin{array}{rccl} \phi : & \mathbb{R} \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (s, x_1, x_2) & \longmapsto & \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{array} \tag{3.26}$$

We choose $N = 600$, $Q_0 = (1, 2)$ and $Q_N = (2, 1)$. Let F and G denote:

$$F = \frac{\partial \phi}{\partial s}(0, Q) = (-Q^2, Q^1) \quad \text{and} \quad G = \frac{\partial L}{\partial v}(Q, \Delta^\alpha_- Q, \tau) = (\Delta^\alpha_- Q^1, \Delta^\alpha_- Q^2). \tag{3.27}$$

The computation of (EL^α_h) gives the following graphics:



(a) Solution $Q = (Q^1, Q^2)$ of (EL^α_h)

(b) Quantity $\sum_{r=1}^{N-1} \alpha_r A_r\big(F \cdot \sigma^r(G)\big)$

Figure 1: Computation of (EL^α_h)

- Figure 1a represents the discrete solutions of (EL^α_h)
- and Figure 1b represents the explicit computable quantity $\sum_{r=1}^{N-1} \alpha_r A_r\big(F \cdot \sigma^r(G)\big)$. As expected from Theorem 10, this quantity is a constant of motion of (EL^α_h).

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